Formulae to Solve Polynomial Equations

The Linear Formula

$$x = \frac{-b}{a}$$

The linear formula gives the solution of $ax + b = 0$ for real numbers a, b with $a \neq 0$.

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula gives the solutions of $ax^2 + bx + c = 0$ for real numbers a, b, c with $a \neq 0$.

The Cubic Formula

$$x = \frac{-2b + \left(\frac{-1+\sqrt{-3}}{2}\right)^n \sqrt[3]{4\left(-2b^3 + 9abc - 27a^2d + \sqrt{(-2b^3 + 9abc - 27a^2d)^2 - 4(b^2 - 3ac)^3}\right)} + \left(\frac{-1-\sqrt{-3}}{2}\right)^n \sqrt[3]{4\left(-2b^3 + 9abc - 27a^2d - \sqrt{(-2b^3 + 9abc - 27a^2d)^2 - 4(b^2 - 3ac)^3}\right)}}{6a}$$

The cubic formula gives the solutions of $ax^3 + bx^2 + cx + d = 0$ for real numbers a, b, c, d with $a \neq 0$.

Directions: Take $n = 0, 1, 2$. Use real cube roots if possible, and principal roots otherwise.

The Quartic Formula

$$x = \frac{-3b \pm \left(\sqrt{3\left(3b^2 - 8ac + 2a\sqrt[3]{4\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3}\right)} + 2a\sqrt[3]{4\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace - \sqrt{(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3}\right)}\right)} \pm \sqrt{3\left(3b^2 - 8ac + 2a\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{4\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3}\right)} + 2a\left(\frac{-1-\sqrt{-3}}{2}\right)\sqrt[3]{4\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace - \sqrt{(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3}\right)}\right)}\right) \pm \operatorname{sgn}\left(\left(\operatorname{sgn}\left(-b^3 + 4abc - 8a^2d\right) - \frac{1}{2}\right)\left(\operatorname{sgn}\left(\max((2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3, \min(3b^2 - 8ac, 3b^4 + 16a^2c^2 + 16a^2bd - 16ab^2c - 64a^3e))\right) - \frac{1}{2}\right)\right)\sqrt{3\left(3b^2 - 8ac + 2a\left(\frac{-1-\sqrt{-3}}{2}\right)\sqrt[3]{4\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3}\right)} + 2a\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{4\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace - \sqrt{(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2 - 4(c^2 - 3bd + 12ae)^3}\right)}\right)}}{12a}$$

The quartic formula gives the solutions of $ax^4 + bx^3 + cx^2 + dx + e = 0$ for real numbers a, b, c, d, e with $a \neq 0$.

Directions: Choose all possibilities for the three $\pm$ signs with the last two equivalent. Use real cube roots if possible, and principal roots otherwise.